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Bargmann representation for Landau levels in two dimensions

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Abstract

We present a formulation of the quantum mechanics of an electron gas confined to two dimensions in a strong magnetic field within the framework of the Hilbert space of analytic functions (Bargmann's space). Our approach extends the representation introduced by Girvin and Jach for the ground state to arbitrary Landau levels and to the regime of coupling between Landau levels. By projecting out the rapid cyclotron motion, the quantum mechanics of the slow guiding centre motion is converted into a system of coupled-channel equations describing the coupling between Landau levels due to the (disorder) potentials. In the limit of strong fields, the coupled-channel equations can be solved perturbatively. For the single-channel case we derive a WKB-like quantization condition for the one-dimensional motion along equipotential lines for arbitrary Landau levels. Two applications of this formalism are discussed: the weak-levitation problem in quantum Hall systems and a two-electron quantum dot in a strong magnetic field.

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1. Introduction

Electrons confined to two dimensions and subject to a perpendicular homogeneous magnetic field B exhibit a large variety of phenomena. Among the most complex and fascinating ones are the integer and the fractional quantum Hall effects, for which the understanding of the characteristics of electronic wavefunctions of a disordered system in strong magnetic fields is important. Experimental advances made it possible to observe quantum Hall plateau transitions by decreasing the magnetic field strength [1–3] and to get access to new phenomena related to the disappearance of the quantum Hall effect in low magnetic fields. Smaller magnetic fields imply occupation of higher Landau levels provided that the total electron density is fixed. While in the limit of strong fields, extended states exist at discrete energies near the centre

of each disorder-broadened Landau level, in the limit $B = 0$ all single-electron states are expected to be localized. An intriguing scenario, referred to as 'levitation', has been proposed to interpolate between these two limits as B decreases [4, 5]. A description of this transition requires the treatment of coupling between Landau levels.

The classical motion of an electron in two dimensions in the presence of a perpendicular strong magnetic field and a smooth random potential is characterized by two separate time scales. The rapid cyclotron period determines the short time scale proportional to $\frac{1}{B}$. The slow motion corresponds to the drift motion of the electron's guiding centre along equipotential lines and its time scale is proportional to B. For an arbitrary potential $V(x, y)$, the separation of the cyclotron motion from the drift motion in the dynamics governed by the Hamiltonian

$$\hat{H} = \hat{H}_L + V(x, y) = \frac{1}{2m_e}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}(x, y)\right)^2 + V(x, y) \tag{1}$$

can only be performed approximately by adiabatic averaging over the fast motion. In turn, the quantum equivalent is the projection onto a specific Landau level. In the absence of the potential V, the eigenvalues of the Schrödinger equation in a homogeneous magnetic field are the discrete, highly degenerate Landau levels [6] $E_n = \hbar\omega_c\left(n + \frac{1}{2}\right)$, where $n = 0, 1, 2, \ldots$, and $\omega_c = |eB|/(m_e c)$ denotes the cyclotron frequency. Adding a small potential V lifts the degeneracy of the Landau levels and broadened bands develop which remain well separated in the limit of very high B so that coupling between them can be neglected. As B is decreased, this simple picture breaks down.

Separation of the fast cyclotron motion from the slow guiding field motion by projection techniques was pioneered by Girvin and Jach [8] for the ground state. Wavefunctions Ψ_0 restricted to the lowest Landau level are of the form

$$\Psi_0(z, \bar{z}) = \mathrm{e}^{-\frac{z\bar{z}}{4l_B^2}} f(z) \tag{2}$$

where $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$, $l_B = [\hbar c/|eB|]^{-1/2}$ denotes the magnetic length and $f(z)$ is an entire analytic function. The Hilbert space containing $f(z)$ is referred to as the Bargmann space [7]. Using equation (2) the two-dimensional Schrödinger equation for the Hamiltonian of equation (1)

$$\hat{H}\Psi(x, y) = E\Psi(x, y) \tag{3}$$

can be transformed into a complex differential equation for $f(z)$. The extension of this formalism to arbitrary Landau levels is one aim of the present paper and will be discussed in section 2, where we introduce a systematic formalism for projecting equation (3) onto an arbitrary Landau level. The wavefunction in higher Landau levels does not have the simple form of equation (2) but, as will be shown, can be determined completely by a single analytic function. This makes it possible to formulate an effective complex Schrödinger equation for higher Landau levels analogous to Girvin and Jach's equation for the lowest Landau level. Moreover, by projecting onto subspaces of a set of Landau levels, the two-dimensional Schrödinger equation can be converted into a system of coupled-channel equations representing the coupling of different Landau levels due to the potential $V(x, y)$. For strong magnetic fields, analytic solutions of these coupled-channel equations can be generated through a perturbative expansion in inverse powers of B, or equivalently, in powers of the magnetic length squared l_B^2. Various approximation methods will be discussed in section 3. Within a single channel treatment, we derive a WKB-like description for the quantization conditions of the guiding centre motion.

As a further application we consider in section 4.1 the weak levitation problem for the integer quantum Hall effect. We show that the effective channel potential taking into account the coupling to other Landau levels is closely related to but not identical to the renormalized

potential introduced by Haldane and Yang [9] and provides the leading-order contribution to weak levitation.

In section 4.2 we apply the present formulation to calculate the energy spectrum of a two-electron quantum dot, where electron–electron correlations become important. For a harmonic potential $V(x, y)$ the two-electron Schrödinger equation in a homogeneous magnetic field is separable. This allows the validity of the perturbative expansion in powers of the inverse magnetic field strength for a Coulomb interacting system to be tested.

A few technical details will be given in the appendix. We use atomic units ($\hbar = |e| = |m_e| = 1$) throughout.

2. Hilbert space of analytic functions for the Landau problem

2.1. Single Landau level

We consider in the following the two-dimensional Schrödinger equation for a non-interacting electron gas in a strong homogeneous magnetic field (with the Hamiltonian of equation (1)) for which we use in the following the symmetric gauge

$$\vec{A}(x, y) = -\tfrac{1}{2}(\vec{r} \times \vec{B}). \tag{4}$$

The static potential $V(x, y)$ is assumed to possess a convergent power series expansion in x and y or, equivalently, in z and its complex conjugate $\bar{z}$, but is otherwise arbitrary. Following Girvin and Jach [8], we map the kets $|\Psi\rangle$ onto the Hilbert space of entire analytic functions (the Bargmann space [7]) f with the scalar product

$$\langle f|g\rangle = \int \mathrm{d}\mu(z)\overline{f(z)}g(z) \tag{5}$$

and measure

$$\mathrm{d}\mu(z) = \frac{1}{2\pi l_B^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{-|z|^2/2l_B^2}. \tag{6}$$

Multiplying $\bar{z}$ with an element of the Bargmann space, $\bar{z}g(z)$, yields a ket $|\bar{z}g\rangle$ that is not an element of the Bargmann space. Nevertheless, the scalar product $\langle f|\bar{z}g\rangle$ is well defined because of

$$\langle f|\bar{z}g\rangle = \langle zf|g\rangle = \left\langle f\left|2l_B^2\frac{\mathrm{d}}{\mathrm{d}z}g\right.\right\rangle \tag{7}$$

suggesting the replacement of all complex conjugates $\bar{z}$ by differential operators $2l_B^2\frac{\partial}{\partial z}$ in Bargmann space. In view of the commutator,

$$\left[l_B^2\frac{\mathrm{d}}{\mathrm{d}z}, z\right] = l_B^2 \tag{8}$$

which is equivalent to that of boson ladder operators, the potential function $V(x, y) = V(z, \bar{z})$ becomes now a function of non-commuting operators in the Bargmann space, which requires a prescription for operator ordering. We will show below that our projection approach automatically yields the correct operator ordering which turns out to be normal ordering (in the terminology of [8]) denoted by colons, $:V\left(z, 2l_B^2\frac{\mathrm{d}}{\mathrm{d}z}\right):$, with the understanding that all derivative operators are kept to the left of the complex variable z. We note that in view of the commutation rule (equation (8)) and the correspondence $\left(a^\dagger \leftrightarrow z, a \leftrightarrow \frac{\mathrm{d}}{\mathrm{d}z}\right)$ this would correspond to anti-normal ordering for boson ladder operators in the context of field theory.

Determination of an effective Schrödinger equation for the electronic motion in the nth Landau level involves the projection

$$|\Psi_n\rangle = \hat{P}_n|\Psi\rangle \tag{9}$$

where

$$\hat{P}_n H_L \hat{P}_n = \frac{2n+1}{2}\omega_c \hat{P}_n. \tag{10}$$

Application to equation (3) leads to an effective single-channel Schrödinger equation

$$\hat{P}_n V|\Psi_n\rangle = \left(E - \frac{2n+1}{2}\omega_c\right)|\Psi_n\rangle = E_n|\Psi_n\rangle \tag{11}$$

where we defined $E_n = E - \frac{2n+1}{2}\omega_c$ to simplify notation. Physically, this corresponds to the Schrödinger equation for the motion of the guiding centre upon adiabatic elimination of the (fast) cyclotron motion. Equation (11) is an acceptable approximation only if a sizeable disparity in time scales between the cyclotron motion and the guiding centre motion exists, i.e. the B field is sufficiently large and coupling to other Landau levels can safely be neglected.

Transformation of equation (11) into the Bargmann representation requires an explicit representation of the projector $\hat{P}_n$. The matrix representation of the projection operator can be determined from the residue of the pole of the Green's function in a homogeneous magnetic field [10] pertaining to the energy eigenvalue of the nth Landau level as

$$P_n(z,\bar{z},\xi,\bar{\xi}) = \langle z|P_n|\xi\rangle = \frac{1}{2\pi l_B^2} L_n\left(\frac{1}{2l_B^2}|z-\xi|^2\right) \mathrm{e}^{-\frac{1}{4l_B^2}(z\bar{z}-2z\bar{\xi}+\xi\bar{\xi})} \tag{12}$$

where L_n denotes the Laguerre polynomial of degree n. The projected Schrödinger equation (equation (11)) reads in the position representation

$$\int \mathrm{d}\xi\, \mathrm{d}\bar{\xi}\, P_n(z,\bar{z},\xi,\bar{\xi}) V(\xi,\bar{\xi})\Psi_n(\xi,\bar{\xi}) = E_n \Psi_n(z,\bar{z}). \tag{13}$$

The wavefunction $\Psi_n(z,\bar{z})$ in equation (13) is no longer of the simple form of equation (2). Instead, the wavefunction within the subspace of the nth Landau level has the general form [11]

$$\Psi_n(z,\bar{z}) = (f_{n,0}(z) + f_{n,1}(z)\bar{z} + f_{n,2}(z)\bar{z}^2 + \cdots + f_{n,n}(z)\bar{z}^n)\, \mathrm{e}^{-\frac{1}{4l_B^2}z\bar{z}} \tag{14}$$

where the $f_{n,i}(z)$, $i = 0,\ldots,n$, denote a set of entire analytic functions. The functions $f_{n,i}$ are not independent of each other but can be generated from the single analytic function $f_{n,n}(z) := f_n(z)$. By definition, the wavefunction of the nth Landau level must satisfy the eigenvalue equation of the projector $\hat{P}_n\Psi_n = \Psi_n$. With the explicit form of the matrix representation (equation (13)) of the projection operator and the general form of the wavefunction (equation (14)), this condition yields

$$\sum_{i=0}^{n} \frac{1}{2\pi l_B^2} \int \mathrm{d}\xi\, \mathrm{d}\bar{\xi}\, \mathrm{e}^{-\frac{1}{2l_B^2}\bar{\xi}(\xi-z)} (-1)^i \binom{n}{i} \frac{1}{i!} \left(\frac{1}{2l_B^2}(z-\xi)(\bar{z}-\bar{\xi})\right)^i$$
$$\times (f_{n,0}(\xi) + f_{n,1}(\xi)\bar{\xi} + \cdots + f_{n,n}(\xi)\bar{\xi}^n) = f_{n,0}(z) + \cdots + f_{n,n}(z)\bar{z}^n. \tag{15}$$

Comparing the terms of different order in $\bar{z}$ yields a system of $(n+1)$-coupled differential equations for $m = 0,\ldots,n$

$$\sum_{i=m}^{n} \frac{1}{2\pi l_B^2} \int \mathrm{d}\xi\, \mathrm{d}\bar{\xi}\, \mathrm{e}^{-\frac{1}{2l_B^2}\bar{\xi}(\xi-z)} \frac{(-1)^i}{i!} \binom{n}{i} \left(\frac{1}{2l_B^2}\right)^i (z-\xi)^i \binom{i}{m} (-\bar{\xi})^{i-m}$$
$$\times (f_{n,0}(\xi) + \cdots + f_{n,n}(\xi)\bar{\xi}^n) = f_{n,m}(z). \tag{16}$$

The factor $\mathrm{e}^{-\frac{1}{2l_B^2}\bar{\xi}(\xi-z)}$ in equation (16) is the so-called principal vector which plays the role of the Dirac $\delta(z-\xi)$ function in Bargmann space [7]. In the appendix we prove the important equality (a generalization of Bargmann's integral equation (A.2))

$$\frac{1}{2\pi l_B^2} \int \mathrm{d}\xi\, \mathrm{d}\bar{\xi}\, \mathrm{e}^{-\frac{1}{2l_B^2}\bar{\xi}(\xi-z)} V(\xi,\bar{\xi}) F(\xi,z) =: V\left(\xi, 2l_B^2 \frac{\mathrm{d}}{\mathrm{d}\xi}\right): F(\xi,z)|_{\xi=z} \tag{17}$$

which enables us to transform the integral equation (16) into the following differential equation:

$$\sum_{i=m}^{n} \frac{(-1)^m n!}{i!(n-i)!m!(i-m)!} \sum_{l=0}^{n} \left(2l_B^2\right)^{-m+l} \left[\left(\frac{\partial}{\partial \xi}\right)^{i-m+l} (z-\xi)^i f_{n,l}(\xi) \right]_{\xi=z} = f_{n,m}. \tag{18}$$

Further algebraic manipulations lead to the following recursion formula for $f_{n,m}(z)$,

$$\sum_{i=m}^{n} \binom{n}{i} \binom{i}{m} (-1)^{i+m} \sum_{l=m+1}^{n} \left(2l_B^2\right)^{l-m} \binom{i-m+l}{i} \left(\frac{\partial}{\partial z}\right)^{l-m} f_{n,l}(z) = f_{n,m}(z) \tag{19}$$

with initial value $f_{n,n}(z) =: f_n(z)$ and $m = 0, 1, \ldots, n-1$. It can be proved by complete induction that the functions $f_{n,m}$ are explicitly given by

$$f_{n,m} = \left(-2l_B^2\right)^{n-m} \binom{n}{m} \left(\frac{\partial}{\partial z}\right)^{n-m} f_n(z) \qquad m = 0, 1, \ldots, n-1. \tag{20}$$

Consequently, the generalization of equation (2) reads

$$\Psi_n(z, \bar{z}) = \mathrm{e}^{-\frac{1}{4l_B^2} z\bar{z}} \sum_{m=0}^{n} \left(-2l_B^2\right)^{n-m} \binom{n}{m} \left(\frac{\partial}{\partial z}\right)^{n-m} f_n(z) \bar{z}^m. \tag{21}$$

Similarly, the Bargmann representation of the channel potential takes on a more complicated structure and depends on the channel index n. In order to determine its explicit form, we insert equation (21) into equation (13) which leads to

$$\frac{1}{2\pi l_B^2} \int \mathrm{d}\xi \, \mathrm{d}\bar{\xi} \, \mathrm{e}^{-\frac{1}{2l_B^2}\bar{\xi}(\xi - z)} \frac{(-1)^n}{n!} \left(\frac{1}{2l_B^2}(z-\xi)\right)^n V(\xi, \bar{\xi}) \sum_{m=0}^{n} \left(-2l_B^2\right)^m \binom{n}{m} \frac{\partial^m f_n(\xi)}{\partial \xi^m} \bar{\xi}^{n-m}$$
$$= E_n f_n(z). \tag{22}$$

Employing now our assumption that the potential $V(z, \bar{z})$ can be expanded in a series of positive powers of z and $\bar{z}$, this integral equation can be transformed into a differential equation for the Bargmann state $f_n(z)$, in analogy to equation (16), by applying equation (A.4)

$$\frac{(-1)^n}{n!} \sum_{m=0}^{n} (-1)^m \binom{n}{m} : \left(\frac{\partial}{\partial \xi}\right)^{n-m} V\left(\xi, 2l_B^2 \frac{\partial}{\partial \xi}\right) : (z-\xi)^n \left. \frac{\partial^m f_n(\xi)}{\partial \xi^m} \right|_{\xi=z} = E_n f_n(z). \tag{23}$$

In the special case of the lowest Landau level $n = 0$, equation (23) reduces to

$$: V\left(z, 2l_B^2 \frac{\partial}{\partial z}\right) : f_0(z) = \left(E - \frac{\omega_c}{2}\right) f_0(z) \tag{24}$$

which agrees with the result first given by Girvin and Jach [8]. For an arbitrary Landau level n, equation (23) can be written in analogy to equation (24) as

$$: V_n\left(z, 2l_B^2 \frac{\partial}{\partial z}\right) : f_n(z) = \left(E - \frac{2n+1}{2}\omega_c\right) f_n(z) = E_n f_n(z). \tag{25}$$

$V_n(z, \bar{z})$ denotes the effective potential for the nth Landau level and can be proved by the method of complete induction to be

$$V_n(z, \bar{z}) = \sum_{m=0}^{n} \left(2l_B^2\right)^m \binom{n}{m} \frac{1}{m!} \frac{\partial^{2m} V}{\partial z^m \partial \bar{z}^m}. \tag{26}$$

Equation (25) plays the role of an effective Born–Oppenheimer-like Schrödinger equation for the adiabatic motion of the guiding centre for an arbitrary Landau level when the coupling to other Landau channels can be neglected. The cyclotron motion is thereby adiabatically eliminated. The Schrödinger equation in two dimensions is reduced, after projection onto a

single Landau level, to a quasi one-dimensional problem of the motion along equipotential lines. The two position coordinates of the electron are replaced by the coordinates of its guiding centre which, however, do not commute. The important point to be noted is that the adiabatic channel potential (equation (26)) agrees with the original bare potential only for the lowest Landau level (equation (24)), while it contains magnetic-field-dependent corrections for higher Landau levels.

2.2. Subspace of several Landau levels

Mixing effects between Landau levels correspond to non-adiabatic corrections (or non-Born–Oppenheimer effects in the language of molecular dynamics) to the effective Schrödinger equation (equation (25)). They can be described in terms of coupled-channel equations describing the dynamical coupling between different adiabatic channels due to the coupling by the potential V. The coupled-channel equations can be derived by projecting equation (3) onto a subspace $(0, n)$ spanned by the Landau levels $i = 0, \ldots, n$

$$\hat{P}_{(0,n)} = \sum_{i=0}^{n} \hat{P}_i. \tag{27}$$

Accordingly,

$$\hat{P}_{(0,n)}|\Psi\rangle := \sum_{i=0}^{n} \hat{P}_i|\Psi\rangle = \sum_{i=0}^{n} |\Psi_i\rangle. \tag{28}$$

With equations (21), (27) and (12), the projected Schrödinger equation for $i = 0, 1, \ldots, n$

$$\sum_{j=0}^{n} \hat{P}_i V |\Psi_j\rangle = E_i |\Psi_i\rangle \tag{29}$$

reads in the position representation

$$\frac{1}{2\pi l_B^2} \int \mathrm{d}\xi\, \mathrm{d}\bar{\xi} L_i\left(\frac{|z-\xi|^2}{2l_B^2}\right) \mathrm{e}^{\frac{-\bar{\xi}(\xi-z)}{2l_B^2}} \sum_{j=0}^{n}\sum_{m=0}^{j} \binom{j}{m} \left(-2l_B^2 \frac{\partial}{\partial \xi}\right)^{j-m} f_j(\xi)\bar{\xi}^m V(\xi, \bar{\xi})$$

$$= E_i \sum_{m=0}^{i} \left(-2l_B^2\right)^{i-m} \binom{i}{m} \left(\frac{\partial}{\partial z}\right)^{i-m} f_i(z)\bar{z}^m. \tag{30}$$

Using equation (A.4), this system of coupled integral equations can be transformed, similar to the previous case of a single Landau level, into a system of coupled differential equations. For simplicity, we give the result only for the three-dimensional subspace spanned by the lowest Landau levels $n = 0$, 1 and 2. The matrix representation of the system of coupled differential equations (equation (30)) reads

$$\left(\begin{array}{c|c|c} :V: + \frac{\omega_c}{2} & 2l_B^2 :\frac{\partial V}{\partial z}: & 4l_B^4 :\frac{\partial^2 V}{\partial z^2}: \\ \hline :\frac{\partial V}{\partial \bar{z}}: & 2l_B^2 :\frac{\partial^2 V}{\partial \bar{z}\partial z}: + :V: + \frac{3\omega_c}{2} & 4l_B^4 :\frac{\partial^3 V}{\partial \bar{z}\partial z^2}: + 4l_B^2 :\frac{\partial V}{\partial z}: \\ \hline \frac{1}{2} :\frac{\partial^2 V}{\partial \bar{z}^2}: & l_B^2 :\frac{\partial^3 V}{\partial \bar{z}^2 \partial z}: + :\frac{\partial V}{\partial \bar{z}}: & 2l_B^4 :\frac{\partial^4 V}{\partial \bar{z}^2 \partial z^2}: + 4l_B^2 :\frac{\partial^2 V}{\partial \bar{z}\partial z}: + :V: + \frac{5\omega_c}{2} \end{array}\right) \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} = E \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix}. \tag{31}$$

Extensions to higher dimensional subspaces are straightforward but increasingly cumbersome. The diagonal elements of equation (31) correspond to the adiabatic single-channel Schrödinger equations (equation (25)) for the one-dimensional motion of the guiding centre on equipotential lines. The off-diagonal elements describe the coupling between different adiabatic channels, i.e. the coupling between the cyclotron motion and the guiding centre motion, thereby restoring

the full dimensionality of the original problem. Equation (31) closely resembles a system of coupled-channel equations in scattering theory. One important observation connected with equation (31) is that the coupling strength between the ith and the jth Landau level scales with the magnetic length as $l_B^{2|i-j|}$, i.e. rapidly decreases for levels farther apart in the high-field regime ($l_B \to 0$). In general, equation (31) can be solved only numerically. However, because of the rapid fall-off of the off-diagonal elements approximate solutions can be generated perturbatively in the limit of large B fields, which we will discuss in the following section. For the particular case of a polynomial potential such as harmonic and anharmonic potentials or saddle point potentials, the off-diagonal matrix elements for $|i-j| > m$ vanish, where m denotes the degree of the polynomial. The matrix reduces to a banded matrix, and the problem becomes formally equivalent to a tight-binding problem.

3. Approximate solutions

3.1. Semiclassical quantization of the single-channel equation

In the limit of large magnetic fields, the mixing of different Landau levels can be ignored. All off-diagonal elements in equation (31) can be neglected, and the problem is reduced to solving the decoupled one-dimensional Schrödinger equation (25). The magnetic length l_B becomes small compared to any other scale of the problem; therefore, l_B^2 can take the role of an effective $\hbar \to 0$ in a WKB-like 'semiclassical' description. Voros [12] developed a conventional WKB approximation in the Bargmann representation with $z = (x - \mathrm{i}p)/\sqrt{2}$ and $\hbar$ as expansion parameter. The present strong-field WKB approach is different in that the Hamiltonian associated with the kinetic energy of a particle has been adiabatically eliminated in equations (25) and (31). The de Broglie wavelength λ no longer provides the characteristic scale of the problem but is replaced by l_B. The dimensionless order parameter of our expansion will be l_B/L, where L is the characteristic length over which the potential V changes by an amount of the order of the magnetic energy $\hbar\omega_c$. The high magnetic field limit in the single Landau level approximation will become valid when $l_B/L \ll 1$, in other words when the potential landscape changes slowly on the scale of the magnetic length. As in the WKB method, we rewrite the wavefunction as $f_n(z) = \exp\left[\mathrm{i}S(z)\big/l_B^2\right]$ in terms of an 'action' $S(z)$, which we expand in a power series of l_B^2

$$S(z) = S_0(z) + \frac{l_B^2}{i}S_1(z) + \left(\frac{l_B^2}{i}\right)^2 S_2(z) + \cdots. \tag{32}$$

In order to determine the operation of $:V_n\left(z, 2l_B^2\frac{\partial}{\partial z}\right):$ on the wavefunction $f_n(z)$, we employ the Fourier transform of the potential $\tilde{V}_n(k,l) = \int \mathrm{d}x \int \mathrm{d}y V_n(x,y)\exp[-\mathrm{i}(kx+ly)]$ with respect to the real space variables $x = \frac{1}{2}(z+\bar{z})$ and $y = \frac{1}{2\mathrm{i}}(z-\bar{z})$, so that

$$\begin{aligned} :V_n\left(z, 2l_B^2\frac{\partial}{\partial z}\right):&= \frac{1}{2\pi}\int \mathrm{d}k\,\mathrm{d}l\;\tilde{V}_n(k,l):\mathrm{e}^{\mathrm{i}k(z+2l_B^2\partial z)/2}\,\mathrm{e}^{l(z-2l_B^2\partial z)/2}: \\ &= \frac{1}{2\pi}\int \mathrm{d}k\,\mathrm{d}l\;\tilde{V}_n(k,l)\,\mathrm{e}^{(\mathrm{i}k-l)l_B^2\partial z}\,\mathrm{e}^{(\mathrm{i}\frac{k}{2}+\frac{l}{2})z}. \end{aligned} \tag{33}$$

Inserting the WKB ansatz equations (32) and (33) into the Schrödinger equation of the nth Landau level (equation (25)) and expanding the exponential function leads to

$$\begin{aligned} &\frac{1}{2\pi}\int \mathrm{d}k\,\mathrm{d}l\;\tilde{V}_n(k,l)\left[1 + \left(\mathrm{i}kl_B^2 - ll_B^2\right)\partial_z + \frac{1}{2}\left(\mathrm{i}kl_B^2 - ll_B^2\right)^2\partial_z^2 + \cdots\right] \\ &\quad\times\left[\mathrm{e}^{\mathrm{i}\frac{k}{2}z+\frac{l}{2}z}\,\mathrm{e}^{\frac{\mathrm{i}}{l_B^2}S_0(z)+S_1(z)+\frac{l_B^2}{\mathrm{i}}S_2(z)+\cdots}\right] = E_n\,\mathrm{e}^{\frac{\mathrm{i}}{l_B^2}S_0(z)+S_1(z)+\frac{l_B^2}{\mathrm{i}}S_2(z)+\cdots}. \end{aligned} \tag{34}$$

After taking the derivatives we find to zeroth order in l_B^2 (this requires the summation of all terms of the order $O(1)$)

$$\frac{1}{2\pi}\int \mathrm{d}k\,\mathrm{d}l\;\tilde{V}_n(k,l)\,\mathrm{e}^{\mathrm{i}\frac{k}{2}z+\frac{l}{2}z}\,\mathrm{e}^{(\mathrm{i}k-l)iS_0'} = E_n. \tag{35}$$

This equation can be transformed to

$$V_n\left(x=\left(\frac{z}{2}+\mathrm{i}S_0'\right),\,y=\left(-\mathrm{i}\frac{z}{2}-S_0'\right)\right) = V_n(z,\bar{z}=2\mathrm{i}S_0'(z)) = E_n. \tag{36}$$

Equation (36) defines equipotential lines of energy $E_n = E - \omega_c\left(n+\frac{1}{2}\right)$. For $S_0'(z)$ we, therefore, find

$$S_0'(z) = \frac{1}{2\mathrm{i}}\bar{z}_E(z) \tag{37}$$

where $\bar{z}_E(z)$ is implicitly defined by the equation

$$V_n(z,\bar{z}_E(z)) = E_n. \tag{38}$$

Leading-order l_B^2 corrections are represented by the first-order corrections $S_1(z)$. After some straightforward manipulations we find for $S_1(z)$ the differential equation

$$S_1' = \left(\frac{\partial V_n}{\partial\bar{z}}(z,2\mathrm{i}S_0')\right)^{-1}\cdot\left(-\frac{\partial^2 V_n}{\partial z\partial\bar{z}}(z,2\mathrm{i}S_0') - \mathrm{i}S_0''(z)\frac{\partial^2 V_n}{\partial\bar{z}^2}(z,2\mathrm{i}S_0')\right). \tag{39}$$

The domains on which the differential equations for S_0' and S_1' (equations (37) and (39)) are defined are the equipotential lines of V pertaining to fixed energy, i.e. are one-dimensional manifolds in the two-dimensional configuration space. If we now require that V, in addition to the expansion in a series of positive powers of z and $\bar{z}$, possesses a non-vanishing gradient everywhere on the equipotential line, S_0' and S_1' are holomorphic functions on a strip containing the equipotential line. A non-vanishing gradient implies that on the equipotential line the potential V does not have an extremum or a saddle. In particular, equipotential lines form closed curves which do not intersect with themselves. Unlike the Schrödinger representation, where quantization is achieved by imposing square-integrability of the wavefunction, in the Bargmann representation it is the analyticity of the wavefunction which determines the eigenvalues [12, 13]. In particular, the WKB wavefunction has to be single valued on the equipotential curve. Therefore, the phase of the wavefunction has to vary by multiples of 2π when we complete a loop around the closed potential line

$$\oint \mathrm{d}\,\mathrm{Im}[\ln f(z)] = \mathrm{Im}\left(\oint\frac{f'(z)}{f(z)}\,\mathrm{d}z\right) = 2\pi k \qquad k=0,1,2,3\ldots. \tag{40}$$

As an element of Bargmann's space, $f(z)$ has to be an analytic function. The integral $\oint\frac{\mathrm{d}\ln f(z)}{\mathrm{d}z}\,\mathrm{d}z$ yields $2\pi\mathrm{i}$ times the number of zeros of $f(z)$ in the interior of the closed equipotential line [14]. The integral $\oint\frac{\mathrm{d}\ln f(z)}{\mathrm{d}z}\,\mathrm{d}z$, therefore, is purely imaginary and the quantum number k is positive or zero. The quantization condition can be alternatively written as

$$\frac{1}{\mathrm{i}}\oint\frac{f'(z)}{f(z)}\,\mathrm{d}z = \frac{1}{\mathrm{i}}\left[\frac{1}{l_B^2}\oint\left(\mathrm{i}S_0' + l_B^2 S_1'(z)\right)\right]\mathrm{d}z = 2k\pi \qquad k=0,1,2,\ldots. \tag{41}$$

The important point is now that the analytic continuation of S_0' and S_1' beyond the strip containing the equipotential curve may feature poles. The latter give non-zero contributions to the WKB quantization conditions. The first integral, $\frac{1}{l_B^2}\oint S_0'(z)\,\mathrm{d}z = \frac{1}{2l_B^2}\oint\bar{z}_E(z)\,\mathrm{d}z$, measures the area enclosed by the equipotential curve in units of l_B^2, in analogy to the classical action (the phase-space area in units of $\hbar$) in the ordinary WKB approach. It should be noted

that in the Bargmann representation with normal operator ordering, the second integral $-\mathrm{i}\oint S_1'(z)\,\mathrm{d}z$ generally does not reduce to a winding number as is the case for symmetric (Weyl) operator ordering, where $-\mathrm{i}\oint S_1'(z)\,\mathrm{d}z = \frac{\mathrm{i}}{2}\oint \mathrm{d}\big(\ln\frac{\partial V_n}{\partial \bar{z}}\big)$ [12]. In the latter case the winding number of the function $\frac{\partial V_n}{\partial \bar{z}}$ completely accounts for the Maslov correction [12, 13]. The key point is that identification of the Maslov corrections is different for different operator ordering prescriptions. The transformation from symmetric to normal ordering accounts for part of the Maslov correction, while the residual part stems from the second phase integral $-\mathrm{i}\oint S_1'(z)\,\mathrm{d}z$ [12]. Equation (41) represents, to our knowledge, the first derivation of the quantization condition for the guiding centre motion in the strong B-field limit in the Bargmann representation. Previous approaches of semiclassical quantization of the guiding centre motion [15, 16] used coherent state path integral techniques, leading to similar but not identical quantization conditions.

Let us illustrate the quantization rule with the simple case of a harmonic potential, i.e. $V = \frac{\omega^2}{2} z\bar{z}$. The effective single-channel potential according to equation (26) is $V_n = \frac{\omega^2}{2} z\bar{z} + nl_B^2\omega^2$. From equations (37) and (39) we find

$$S_0'(z) = \frac{1}{2\mathrm{i}z}\left(E - \frac{2n+1}{2}\omega_c - nl_B^2\omega^2\right) \tag{42}$$

$$S_1'(z) = -\frac{1}{z}. \tag{43}$$

Both functions have single poles at $z = 0$, i.e. outside any equipotential line. Therefore the quantization condition (equation (41)) yields

$$E_{n,k} = \frac{2n+1}{2}\omega_c + \frac{\omega^2}{\omega_c}\left(n + k + \frac{\alpha}{4}\right) \qquad \text{with } \alpha = 4. \tag{44}$$

These WKB eigenenergies can be shown to be equal to the expansion of the exact energies of the Landau levels in the presence of an additional isotropic harmonic perturbation ($\propto \omega^2$) to first order in $\left(\frac{\omega}{\omega_c}\right)^2$. The wavefunction in Bargmann space pertaining to the energy $E_{n,k}$ is of the form $f_{n,k}(z) = z^k$. The Maslov index of the guiding centre motion counting the number of turning points in two dimensions is $\alpha = 4$: the guiding centre propagates along a circle in the x–y plane, and the velocity components in the x and y directions each change their sign twice on a complete loop.

3.2. High field limit of coupled channel equations

The problem of Landau level mixing was recently analysed by Haldane and Yang [9] within the framework of a renormalized potential V_n^R in a given Landau level n, which takes perturbatively into account coupling effects to other Landau levels. With such a potential, it should be possible to represent within the truncated Hilbert space of Landau level n the effects of the coupling to other levels. Since the present treatment (equation (30)) allows for the direct treatment of channel coupling, it is instructive to compare the two approaches.

In the lowest order perturbation theory, the energy of the system in a given Landau level n can be simply calculated by the matrix element of the potential in the nth Landau level

$$E \simeq \frac{2n+1}{2}\omega_c + \langle n|V|n\rangle \tag{45}$$

where $|n\rangle$ represents the basis state of the Hilbert space belonging to the nth Landau level. The energies calculated by equation (45) are obviously the eigenenergies calculated by the single-channel equation (equation (25)), which can be easily verified by noting that the projection operator onto the nth Landau level is $\hat{P}_n = |n\rangle\langle n|$.

In order to account for mixing effects with other Landau levels, the bare potential V in equation (45) was replaced in [9] by the renormalized potential V_n^R, so that the energy was given by $E \simeq \frac{2n+1}{2}\omega_c + \langle n|V_n^R|n\rangle$. Up to order $O(B^{-3})$, the renormalized potential V_n^R reads in terms of the bare potential V in the complex-coordinate representation

$$V_n^R(z,\bar{z}) = V(z,\bar{z}) - \frac{2l_B^2}{\omega_c}\frac{\partial V}{\partial z}\frac{\partial V}{\partial \bar{z}} + 6\left(n+\frac{1}{2}\right)\frac{l_B^4}{\omega_c}\frac{\partial^2 V}{\partial z^2}\frac{\partial^2 V}{\partial \bar{z}^2} + O\left(l_B^8\right). \quad (46)$$

Since for the single-channel case the energies calculated by equation (45) are equivalent to eigenenergies calculated by equation (25), it is tempting to assume that, in order to account for Landau level coupling effects, the renormalized potential for Landau level n (equation (46)) can serve, upon normal ordering, as the bare potential of the effective Schrödinger equation of the nth Landau level (equation (25)), so that

$$:V_n^{\text{eff}}\left(z, 2l_B^2\frac{\partial}{\partial z}\right): f_n(z) = \left(E - \frac{2n+1}{2}\omega_c\right) f_n(z) \quad (47)$$

where the effective potential including coupling effects is given by

$$V_n^{\text{eff}}(z,\bar{z}) = \sum_{m=0}^{n}\left(2l_B^2\right)^m \binom{n}{m}\frac{1}{m!}\frac{\partial^{2m}V_n^R(z,\bar{z})}{\partial z^m \partial \bar{z}^m}. \quad (48)$$

We will show that the perturbative solution of our system of coupled differential equations (30) leads to precisely the same effective potential (equation (48)), thereby proving the validity of our assumption.

The differential matrix equation (equation (31)) is expanded perturbatively in orders of l_B^2 in order to calculate the effective potential of the Landau levels $n = 0, 1$ including coupling effects to higher Landau levels. Focusing first on $n = 0$, the coupling from higher Landau levels will result in a small admixture of wavefunctions of higher Landau levels $f_i, i = 1, 2$, which will be at least one order of magnitude in l_B^2 smaller than f_0. The first row of the matrix equation (31) reads

$$:V: f_0(z) + 2l_B^2 :\frac{\partial V}{\partial z}: f_1(z) + 4l_B^4 :\frac{\partial^2 V}{\partial z^2}: f_2(z) = \left(E - \frac{\omega_c}{2}\right) f_0(z). \quad (49)$$

To leading order the admixture of f_1 and f_2 are given (according to the second and third rows of equation (31)) by

$$f_1(z) \simeq -l_B^2 :\frac{\partial V}{\partial \bar{z}}: f_0(z) \quad (50)$$

$$f_2(z) \simeq -\frac{l_B^2}{4} :\frac{\partial^2 V}{\partial \bar{z}^2}: f_0(z). \quad (51)$$

Introducing these expressions into equation (49) we get

$$:V: f_0(z) - 2l_B^2 :\frac{\partial V}{\partial z}::\frac{\partial V}{\partial \bar{z}}: f_0(z) - l_B^6 :\frac{\partial^2 V}{\partial z^2}::\frac{\partial^2 V}{\partial \bar{z}^2}: f_0(z) = \left(E - \frac{\omega_c}{2}\right) f_0(z). \quad (52)$$

In equation (52) the operator product of two normal-ordered operators appears. The operator product $:A\left(z, 2l_B^2\partial z\right)::B\left(z, 2l_B^2\partial z\right):$ is in general not equal to $:AB\left(z, 2l_B^2\partial z\right):$. The difference can be calculated straightforwardly in expanding A and B in a power series of z and $\bar{z}$ and reads

$$:A::B: = :AB: - 2l_B^2 :\frac{\partial A}{\partial z}\frac{\partial B}{\partial \bar{z}}: + O\left(l_B^4\right). \quad (53)$$

Applying equation (53) to the operator product $:\frac{\partial V}{\partial z}::\frac{\partial V}{\partial \bar{z}}:$ and noting that $:\frac{\partial^2 V}{\partial z^2}::\frac{\partial^2 V}{\partial \bar{z}^2}: = :\frac{\partial^2 V}{\partial z^2}\frac{\partial^2 V}{\partial \bar{z}^2}: + O\left(l_B^2\right)$ equation (52) becomes

$$:V: f_0(z) - 2l_B^4 :\frac{\partial V}{\partial z}\frac{\partial V}{\partial \bar{z}}: f_0(z) + 3l_B^6 :\frac{\partial^2 V}{\partial z^2}\frac{\partial^2 V}{\partial \bar{z}^2}: f_0(z) = \left(E - \frac{\omega_c}{2}\right) f_0(z). \tag{54}$$

The effective potential in the lowest Landau level including coupling effects to higher Landau levels up to order $O\left(l_B^6\right)$ can, therefore, be identified as

$$V_0^{\text{eff}} = V - 2\frac{l_B^2}{\omega_c}\frac{\partial V}{\partial z}\frac{\partial V}{\partial \bar{z}} + 3\frac{l_B^4}{\omega_c}\frac{\partial^2 V}{\partial z^2}\frac{\partial^2 V}{\partial \bar{z}^2}. \tag{55}$$

This effective potential agrees with the renormalized potential equation (46) for $n = 0$. Note that for $n = 0$ no additional correction terms arise from the transformation to the effective channel potential (equation (48)). The extension to the Landau level $n = 1$ is straightforward. To this end we extend the matrix equation (equation (31)) to $n = 3$. Up to order $O\left(l_B^6\right)$ coupling effects the second row of the extended matrix equation (31) yields

$$\begin{aligned} &:\frac{\partial V}{\partial \bar{z}}: f_0 + 2l_B^2 :\frac{\partial^2 V}{\partial \bar{z}\partial z}: f_1 + :V: f_1 + 4l_B^4 :\frac{\partial^3 V}{\partial \bar{z}\partial z^2}: f_2 + 4l_B^2 :\frac{\partial V}{\partial z}: f_2 + 12l_B^4 :\frac{\partial^2 V}{\partial z^2}: f_3 \\ &\qquad = \left(E - \frac{3\omega_c}{2}\right) f_1. \end{aligned} \tag{56}$$

Proceeding along similar lines as above equation (56) yields to $O\left(l_B^6\right)$,

$$:V_1^{\text{eff}}\left(z, 2l_B^2\frac{\partial}{\partial z}\right): f_1 = \left(E - \frac{3}{2}\omega_c\right) f_1 \tag{57}$$

with

$$\begin{aligned} V_1^{\text{eff}} := {} & V + 2l_B^2\frac{\partial^2 V}{\partial \bar{z}\partial z} - 2\frac{l_B^2}{\omega_c}\frac{\partial V}{\partial \bar{z}}\frac{\partial V}{\partial z} \\ & - 4\frac{l_B^4}{\omega_c}\left(\frac{\partial^2 V}{\partial \bar{z}\partial z}\frac{\partial^2 V}{\partial \bar{z}\partial z} + \frac{\partial^3 V}{\partial \bar{z}\partial z^2}\frac{\partial V}{\partial \bar{z}} + \frac{\partial^3 V}{\partial \bar{z}^2\partial z}\frac{\partial V}{\partial z}\right) + 5\frac{l_B^4}{\omega_c}\frac{\partial^2 V}{\partial \bar{z}^2}\frac{\partial^2 V}{\partial z^2} \end{aligned} \tag{58}$$

which is the effective potential in the Landau level $n = 1$ including coupling effects to other Landau levels. Note that equation (58) is not identical to the renormalized potential equation (46). However, using the renormalized potential (equation (46)) as the bare potential in transformation equation (48), complete equivalence is achieved. In other words, for all $n \geqslant 1$ the correction terms resulting from equation (48) are essential in order to establish equivalence.

4. Applications

4.1. The weak levitation of extended states—Landau level mixing effects near the saddle point

We apply the present method to the problem of the weak levitation of extended states in the integral quantum Hall effect. The starting point is the network model of Chalker and Coddington [17]. In this approximation the disorder is modelled by a random pattern of smooth equipotential contours. Different closed random contours of the same energy are separated from each other by potentials that can be locally approximated by saddle point potentials. Within the framework of percolation theory, states become extended when the energy reaches a critical energy E_C of the saddle point. For this energy the coefficient for transmission through the saddle reaches the value $\frac{1}{2}$. We consider therefore the saddle point

(SP) potential $V_{\mathrm{SP}} = -\frac{\Lambda^2}{2}(x^2 - y^2)$, where Λ^2 is a parameter describing the strength of the saddle. In complex coordinates representation, V_{SP} reads

$$V_{\mathrm{SP}} = -\frac{\Lambda^2}{4}(z^2 + \bar{z}^2). \tag{59}$$

V_{SP} serves as the representative of the local disordered potential which is only valid for distances that are within the correlation length d of the potential, i.e.

$$\sqrt{x^2 + y^2} < d \tag{60}$$

and, furthermore, for which the quadratic approximation (equation (59)) holds. The effective potential of the saddle point including mixing effects is according to equation (48) up to order $O\left(l_B^6\right)$ given by

$$V_{n\,\mathrm{SP}}^{\mathrm{eff}} = -\frac{\Lambda^2}{4}(z^2 + \bar{z}^2) - \frac{l_B^4\Lambda^4}{2}\bar{z}z + \frac{l_B^6\Lambda^4}{4}(2n+3). \tag{61}$$

In addition to the bare saddle point potential two terms appear: a term $\propto \bar{z}z$ corresponds to an inverted harmonic oscillator $\left(\sim l_B^4\right)$ which squeezes the saddle potential in the unstable direction, while it widens it in the stable direction, as can be seen from its Cartesian representation

$$V_{n\,\mathrm{SP}}^{\mathrm{eff}} = -\frac{\Lambda^2}{2}\left(1 + l_B^4\Lambda^2\right)x^2 + \frac{\Lambda^2}{2}\left(1 - l_B^4\Lambda^2\right)y^2 + \frac{l_B^6\Lambda^4}{4}(2n+3). \tag{62}$$

The second correction term in equation (62) resulting from the coupling to other Landau levels leads to an upward shift $\propto l_B^6$. In the following, we will show that squeezing together with the positive energy shift of the saddle point of the order l_B^6 results in an upward shift of the critical transmission energy of $\frac{1}{4}l_B^6\Lambda^4(2n+1)$ relative to the Landau energy. This corresponds to a levitation of the energy of extended states to order B^{-3}. The single-channel Schrödinger equation (47) reads for the saddle point potential

$$-\frac{\Lambda^2}{4}z^2 f_n(z) - l_B^4\Lambda^2 f_n'' - l_B^4\Lambda^4 z f_n' + \frac{l_B^6\Lambda^4}{4}(2n-1)f_n = E_n f_n. \tag{63}$$

Using the ansatz $f_n = \tilde{f}_n \exp\left(-\frac{1}{4}\Lambda^2 l_B^2 z^2\right)$ and neglecting terms of the order $O\left(l_B^8\right)$ we get

$$-\frac{\Lambda^2}{4}z^2\tilde{f}_n(z) - l_B^4\Lambda^2\tilde{f}_n'' = \left(E_n - \frac{l_B^6\Lambda^4}{4}(2n+1)\right)\tilde{f}_n. \tag{64}$$

This is the Schrödinger equation for an inverted harmonic oscillator. Therefore, the problem can be reduced to that of the transmission through a quadratic potential barrier along the real axis [18]. Accordingly, the transmission coefficient reaches the critical value $\frac{1}{2}$ at the critical energy E_C given by the zero of the effective eigenvalue (right-hand side of equation (64)),

$$E_C = \frac{2n+1}{2}\omega_c + \frac{l_B^6\Lambda^4}{4}(2n+1). \tag{65}$$

E_C is shifted upwards relative to the centre of the Landau band, $\frac{1}{2}(2n+1)\omega_c$, by an amount of the order of $+\frac{1}{4}l_B^6\Lambda^4(2n+1)$, which represents the dominant contribution to the weak levitation to order $l_B^6 \propto B^{-3}$.

It is, furthermore, instructive to analyse the origin of various contributions to the energy shift for the levitation with the help of the coupled-channel equations (30) in more detail. Because of the quadratic form of the potential, the calculation of the effective channel equations (30) is particularly simple. For the nth channel they read

$$\begin{aligned}-\frac{\Lambda^2}{4}f_{n-2}(z) - l_B^2\Lambda^2 f_{n-1}'(z) - \frac{\Lambda^2}{4}z^2 f_n(z) - l_B^4\Lambda^2 f_n''(z)\\ - l_B^2\Lambda^2(n+1)z f_{n+1}(z) - l_B^4\Lambda^2(n+2)(n+1)f_{n+2}(z) = E_n f_n(z).\end{aligned} \tag{66}$$

Equation (66) represents the nth row of the matrix representation of the system of coupled-channel equations (31). In the special case of the saddle point potential this matrix is banded with a width of $\Delta n = \pm 2$. More generally, coupling strength decreases as $B^{-|\Delta n|}$. The main contribution of level mixing for a Landau level n is, therefore, the coupling to Landau levels $n-2, n-1, n+1$ and $n+2$. f_n implicitly depends on f_{n+3} (f_{n-3}) through f_{n+1} and f_{n+2} (f_{n-1}, f_{n-2}): this will, however, result in corrections of higher order in l_B. The coupling of a given Landau level of index n to its nearest neighbours with quantum numbers $n-1$ and $n+1$ results in the squeezing of the saddle point potential. In addition, coupling to the next-to-nearest neighbours, the Landau levels of the indices $n-2$ and $n+2$, has to be taken into account. Their effect turns out to be an energy shift of the order of l_B^6 of the saddle point energy. We find that in order to calculate the levitation effect, coupling to the four nearest neighbours has to be taken into account. This highlights the fact that it is not necessarily the coupling to the two nearest neighbours which accounts for the leading-order mixing corrections.

4.2. Two electrons in a parabolic quantum dot

In this section, we present one further application that goes beyond the level of the non-interacting electron gas and which addresses the description of electron–electron correlations in the limit of large B-fields. To this end, we consider the problem of two electrons interacting via Coulomb interaction and subject to a homogeneous magnetic field B in the z-direction in the presence of an additional harmonic confining potential $V = \frac{m}{2}\omega^2\vec{r}^2$. This problem of a parabolic quantum dot in a strong magnetic field is exactly solvable due to its separability in centre-of-mass and relative coordinates. By comparing the exact solution with the perturbative expansion in B^{-1} of section 3.2, the convergence of the perturbation series in the presence of electron–electron correlations can be tested.

The Hamiltonian for the two-dimensional system expressed in terms of centre-of-mass coordinates $\vec{R}_{\rm CM} = 1/2(\vec{r}_1 + \vec{r}_2)$ $\left(\vec{r}_i^{\,T} = (x_i, y_i)\right)$ and relative coordinates $\vec{r} = \vec{r}_1 - \vec{r}_2$ reads

$$\hat{H} = \hat{H}_{\rm CM} + \hat{H}_{\rm rel} = \frac{1}{2M}\left[\hat{\vec{p}}_{\rm CM} - \frac{e}{c}\vec{A}_{\rm CM}\right]^2 + \frac{M}{2}\omega^2\vec{R}_{\rm CM}^2 + \frac{1}{2m_{\rm rel}}\left[\hat{\vec{p}}_{\rm rel} - \frac{e}{c}\vec{A}_{\rm rel}\right]^2 + \frac{m_{\rm rel}}{2}\omega^2\vec{r}^2 + \frac{e^2}{|\vec{r}|} \tag{67}$$

where $M = 2m_{\rm eff}, m_{\rm rel} = \frac{m_{\rm eff}}{2}$ ($m_{\rm eff}$ is the effective electron mass in au), $\vec{A}_{\rm CM}^{\,T} = -B_{\rm CM}/2(-Y_{\rm CM}, X_{\rm CM})$, $B_{\rm CM} = 2B$, $\vec{A}_{\rm rel}^{\,T} = -B_{\rm rel}/2(-y, x)$ and $B_{\rm rel} = B/2$. The centre-of-mass part of the Schrödinger equation can be solved exactly by reduction to a two-dimensional harmonic oscillator with the renormalized frequency $\tilde{\omega}_{\rm CM} = \frac{1}{2}\sqrt{4\omega^2 + \omega_{c,\rm CM}^2}$ [25]. The relative motion of the system can also be treated exactly. The magnetic field dependence leads to a renormalized frequency $\tilde{\omega}_{\rm rel} = \frac{1}{2}\sqrt{4\omega^2 + \omega_{c,\rm rel}^2}$. The resulting problem can be solved analytically for a denumerably infinite set of oscillator frequencies [24], while it can be reduced to a one-dimensional quadrature for other frequencies. The potential governing the relative motion reads in the complex representation

$$V_{\rm rel}(z, \bar{z}) = \frac{m_{\rm rel}}{2}\omega^2 z\bar{z} + \frac{1}{\sqrt{z\bar{z}}}. \tag{68}$$

To simplify notation we drop the index 'rel' in the coordinates and in other quantities of the internal system in the following. As discussed in section 3.2 the perturbative approach fixes the smallest characteristic length of the system to be the magnetic length l_B^2. Additional potentials are required to be smooth and slowly varying on that scale. The singularity of the

Coulomb potential as $r \to 0$ violates this constraint. One can, instead, employ a magnetically shielded Coulomb potential

$$V(z,\bar{z}) = \frac{m_{\mathrm{rel}}}{2}\omega^2 z\bar{z} + \frac{1}{\sqrt{z\bar{z} + \alpha l_B^2}} \tag{69}$$

in analogy to atomic systems in strong B-fields [21], where the parameter $\alpha \geqslant 0$ controls the effective shielding. For $\alpha = 1$ the Coulomb potential is quenched on the scale of the magnetic length while for $\alpha = 0$ the pure Coulomb potential emerges. For $\alpha > 0$, an exact solution can only be found by numerical quadrature. The size of the shielding length plays a crucial role in controlling the convergence of the perturbation series.

To calculate the effective potential of a given Landau level n taking into account coupling to other levels, V has to be normalized according to equation (46) which yields

$$V_n^R = V_{\mathrm{ee}} + V_A + V_B + V_C + V_D + O(B^{-4}) \tag{70}$$

with

$$V_{\mathrm{ee}} = \frac{1}{\sqrt{z\bar{z} + \alpha l_B^2}} \tag{71}$$

$$V_A = \frac{1}{2}\,\omega^2 z\bar{z}\left(1 - \frac{\omega^2 l_B^2}{\omega_c}\right) \tag{72}$$

$$V_B = \frac{\omega^2 l_B^2}{\omega_c}\,\frac{z\bar{z}}{\left(z\bar{z} + \alpha l_B^2\right)^{3/2}} \tag{73}$$

$$V_C = -\frac{l_B^2}{2\omega_c}\,\frac{z\bar{z}}{\left(z\bar{z} + \alpha l_B^2\right)^{3}} \tag{74}$$

$$V_D = \frac{27}{8}\left(n + \frac{1}{2}\right)\frac{l_B^4}{\omega_c}\,\frac{(z\bar{z})^2}{\left(z\bar{z} + \alpha l_B^2\right)^{5}}. \tag{75}$$

V_{ee} denotes the bare electron–electron interaction (unscreened or screened Coulomb potential). The correction (V_A) due to the coupling to other Landau levels results in a renormalization of the frequency of the harmonic potential, which corresponds to the first-order correction of the expansion of the exact renormalized frequency $\tilde{\omega}_{\mathrm{rel}}$. The second correction (V_B) represents a renormalization of the electronic charge of the Coulomb potential. It emerges from the cross-term between the Coulomb and the harmonic potential. Without the magnetic shielding (equation (69)) higher order corrections would result in terms more singular than the original Coulomb potential and are proportional to r^{-4} and r^{-6}. Consequently energy corrections due to these potentials would be only non-diverging for states with higher angular momenta where a high centrifugal barrier prevents a high density of the wavefunction at small r.

The Schrödinger equation (equation (47)) in complex-coordinate representation should be solved with equation (70) as perturbation. Analytic expressions can be found only for the limiting case $\alpha = 0$ of a pure Coulomb potential. The wavefunctions in the Bargmann representation are given by $f_n^m = z^{n-m}$, where n labels the Landau level and m the eigenvalue m of the angular momentum l_z. Because of the analyticity of the wavefunction in the Bargmann representation we get the constraint $n \geqslant m$. The Landau quantum number n is related to m through $n = n_r + \frac{1}{2}(|m| + m)$, where $n_r \geqslant 0$ denotes the radial quantum number. Each term

V_j of the renormalized potential of equation (70) then gives an energy correction ϵ_j to the Landau energy $\frac{2n+1}{2}\omega_c$ according to

$$:\sum_{i=0}^{n}\left(2l_B^2\right)^i\binom{n}{i}\frac{1}{i!}\frac{\partial^{2i}V_j(z,\bar{z})}{\partial z^i\partial\bar{z}^i}\bigg|_{\bar{z}=2l_B^2\partial z}:z^{n-m}=\epsilon_j z^{n-m}\qquad j=A,B,C,\ldots\quad n\geqslant m. \tag{76}$$

We find

$$\epsilon_A=\frac{\omega^2}{\omega_c}\left(1-\frac{\omega^2}{\omega_c^2}\right)(2n-m+1) \tag{77}$$

$$\epsilon_{ee}+\epsilon_B=\left(1+\frac{\omega^2 l_B^2}{\omega_c}\right)\frac{\left(2l_B^2\right)^{-1/2}}{\pi}\sum_{j=0}^{n_r}\frac{\Gamma\left(|m|+j+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(n_r-j+\frac{1}{2}\right)}{j!(n_r-j)!(|m|+j)!} \tag{78}$$

$$\epsilon_C=-\frac{m_{\rm rel}}{8}\frac{(|m|-2)!(2n-m+1)}{(|m|+1)!}|m|\geqslant 2 \tag{79}$$

$$\epsilon_D=27m_{\rm rel}\frac{(2n+1)}{2^8}\frac{(n_r+2)!(|m|-3)!}{|m|!n_r!}\left[1-\frac{2n_r(|m|-2)}{(|m|+1)(n_r+2)}+\frac{2n_r(n_r-1)(|m|-2)(|m|-1)}{(|m|+1)(|m|+2)(n_r+2)(n_r+1)}\right]\qquad |m|\geqslant 3. \tag{80}$$

Note that the corrections ϵ_C and ϵ_D are independent of B. Because of their singular behaviour they give in the pure Coulomb case a finite contribution only for higher angular momenta.

These analytic energy contributions for the Coulomb potential can be compared to the numerically determined ones for the screened Coulomb potential. Only for magnetic energies ω_c higher than the energies of the Coulomb potential $\frac{1}{\epsilon a_{\rm eff}}$ (where $a_{\rm eff}$ is the effective Bohr radius and ϵ is the dielectric constant) and the confining potential of the quantum dot ω is the perturbative expansion expected to converge. We set the scale of the magnetic field B_0 through the condition $\omega_c=\frac{1}{\epsilon a_{\rm eff}}$ such that for $B>B_0$ the external potential can be considered a perturbation of the Landau levels. For GaAs semiconductor microstructures with $m_{\rm rel}=0.035$ and $\epsilon=13$ we get $B_0=3.4\,$T, which is a typical field strength in quantum Hall experiments [26]. The strength of the confining potential we choose at $\omega=0.1$.

Figures 1 and 2 display the relative error of the perturbatively calculated energy levels for $m=-4$ and $n=0$ for the bare (figure 1) and shielded (figure 2) Coulomb potential. In each case we show the error as more terms of the series (equations (76)–(80)) are added. As expected, the relative error approaches zero as $B/B_0\to\infty$. However, the convergence is non-monotonic as a function of the number of terms included. Moreover, the relative error in the case of a screened Coulomb potential is always smaller than the error of the unscreened Coulomb potential at a given B/B_0, as expected from the cut-off of the singularity at $r=0$. For the same reason, for a fixed Landau level the relative error decreases with increasing angular momentum quantum number $|m|$.

For fixed B/B_0 and m, the error increases with increasing Landau quantum number n. The reason for this behaviour is that the radial wavefunction acquires an increasing number of nodes n_r thereby pushing a portion of probability towards $r=0$. This, in turn, increases the weight of the terms V_C and V_D containing high inverse powers in r. Higher order terms beyond V_D are expected to increase the error. We therefore expect the perturbation series to be asymptotic. Nevertheless, we find that for moderate Landau quantum numbers n low-order perturbation leads to excellent agreement with the exact calculation for $B/B_0>1$.

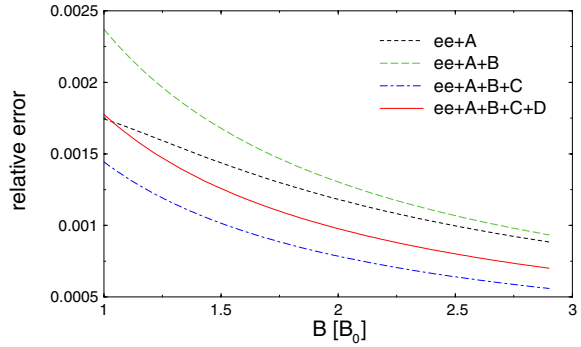

Figure 1. Relative error $\Delta E/E$ of perturbatively calculated energy for the lowest Landau level $n = 0$ and angular momentum $m = -4$ as a function of the magnetic field B in units of B_0 for the bare Coulomb potential ($\alpha = 0$). Shown are sums over different terms of the perturbation series.

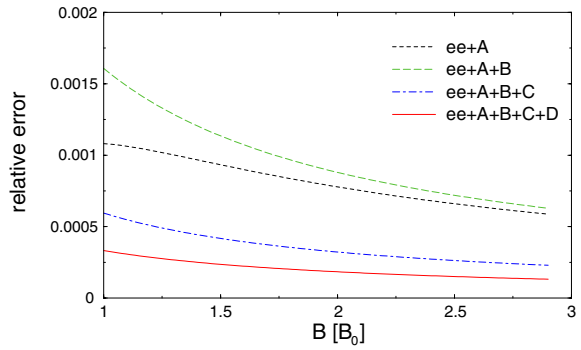

Figure 2. As figure 1, however for a screened Coulomb potential ($\alpha = 1$).

5. Conclusion

We have developed the Bargmann representation of the two-dimensional Schrödinger equation for an electron in a homogeneous perpendicular magnetic field in the presence of sufficiently smooth (disorder) potential for arbitrary Landau levels n. By projection onto the subspace of several Landau levels, an effective Schrödinger equation in terms of a set of coupled channel equations, which describe mixing effects of different Landau levels in this subspace, has been constructed. Projecting onto a single Landau level results in the quasi-one-dimensional Schrödinger equation for the drift motion of the guiding centre in an effective potential which, in general, depends on the Landau level index.

We have shown that the single-channel wavefunction for an arbitrary Landau level of index n is determined by a single analytic function $f_n(z)$. For the lowest Landau level this method reduces to the well-known differential-equation of Girvin and Jach [8]. Furthermore, we have derived the strong-field solution of the single-channel Schrödinger equation in analogy to the conventional WKB approximation. In the present case, the magnetic length l_B rather than the de Broglie wavelength is the expansion parameter. In contrast to conventional semiclassical approaches our energy spectrum includes Maslov corrections from the outset.

We have applied the present formalism to Landau level mixing and levitation. We have shown the close correspondence between the perturbative solution of the coupled-channel

equations and the renormalized potential introduced by Haldane and Yang [9]. We find the dominant weak levitation effects of extended states for the integer quantum Hall effect to be proportional to B^{-3}.

Further we investigated the electron–electron interaction in terms of a perturbative expansion in orders of B^{-1}, for a parabolic two-electron quantum dot in a magnetic field. We compared the perturbative solution with the exact solution of the problem. The perturbative approach is in good agreement with the exact solution for moderate Landau quantum numbers n and angular momenta m in a magnetic field regime accessible in quantum Hall experiments.

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Appendix. Generalization of Bargmann's integral

We consider the integral

$$\frac{1}{2\pi l_B^2}\int \mathrm{d}\xi\,\mathrm{d}\bar{\xi}\,\mathrm{e}^{-\frac{1}{2l_B^2}\bar{\xi}(\xi-z)}V(\xi,\bar{\xi})F(\xi,z)$$
$$:=\frac{1}{2\pi l_B^2}\int_{-\infty}^{\infty}\mathrm{d}\xi_x\int_{-\infty}^{\infty}\mathrm{d}\xi_y\,\mathrm{e}^{-\frac{1}{2l_B^2}(\xi_x-\mathrm{i}\xi_y)[(\xi_x+\mathrm{i}\xi_y)-z]}V(\xi_x,\xi_y)F(\xi_x+\mathrm{i}\xi_y,z) \qquad (A.1)$$

where $\xi=\xi_x+\mathrm{i}\xi_y$, $\bar{\xi}=\xi_x-\mathrm{i}\xi_y$ and z is an arbitrary point in the complex plane. This integral is a generalization of Bargmann's integral [7]

$$\frac{1}{2\pi l_B^2}\int \mathrm{d}\xi\,\mathrm{d}\bar{\xi}\,\mathrm{e}^{-\frac{1}{2l_B^2}\bar{\xi}(\xi-z)}f(\xi)=f(z). \qquad (A.2)$$

We calculate (A.1) for arbitrary potential functions $V(\xi,\bar{\xi})$ with the restriction that they can be expanded in a series of positive powers of ξ and $\bar{\xi}$

$$V(\xi,\bar{\xi})=\sum_{n=0,m=0}a_{nm}\xi^n\bar{\xi}^m. \qquad (A.3)$$

In the following we will assume the function $F(\xi,z)$ to be analytic in the complex variable ξ. We will prove that under these conditions, the generalization of Bargmann's integral equation (A.1) yields

$$\frac{1}{2\pi l_B^2}\int \mathrm{d}\xi\,\mathrm{d}\bar{\xi}\,\mathrm{e}^{-\frac{1}{2l_B^2}\bar{\xi}(\xi-z)}V(\xi,\bar{\xi})F(\xi,z)=:V\left(\xi,2l_B^2\frac{\mathrm{d}}{\mathrm{d}\xi}\right):F(\xi,z)\Big|_{\xi=z}. \qquad (A.4)$$

To prove equation (A.4) we insert the power series expansion of V. Noting that $\bar{\xi}^m\,\mathrm{e}^{-\frac{1}{2l_B^2}\bar{\xi}(\xi-z)}=\left(-2l_B^2\frac{\partial}{\partial\xi}\right)^m\mathrm{e}^{-\frac{1}{2l_B^2}\bar{\xi}(\xi-z)}$ equation (A.1) reads

$$\frac{1}{2\pi l_B^2}\sum_{n,m}a_{nm}\iint \mathrm{d}\xi\,\mathrm{d}\bar{\xi}\,F(\xi,z)\xi^n\bar{\xi}^m\,\mathrm{e}^{-\frac{1}{2l_B^2}\bar{\xi}(\xi-z)}$$
$$=\frac{1}{2\pi l_B^2}\sum_{n,m}a_{nm}\iint \mathrm{d}\xi\,\mathrm{d}\bar{\xi}\,F(\xi,z)\xi^n\left(-2l_B^2\frac{\partial}{\partial\xi}\right)^m\mathrm{e}^{-\frac{1}{2l_B^2}\bar{\xi}(\xi-z)}. \qquad (A.5)$$

After partial integration we get for equation (A.5)

$$\frac{1}{2\pi l_B^2}\sum_{n,m} a_{nm}\iint \mathrm{d}\xi\,\mathrm{d}\bar{\xi}\,\mathrm{e}^{-\frac{1}{2l_B^2}\bar{\xi}(\xi-z)}\left[\left(2l_B^2\frac{\partial}{\partial\xi}\right)^m \xi^n F(\xi,z)\right]. \tag{A.6}$$

The expression $\sum_{n,m} a_{nm}\left(2l_B^2\frac{\partial}{\partial\xi}\right)^m\xi^n = :V\left(\xi, 2l_B^2\frac{\partial}{\partial\xi}\right):$ is the normal-ordered potential, where all derivatives are kept at the left, having replaced the complex variable $\bar{\xi}$ by the operator $2l_B^2\frac{\partial}{\partial\xi}$. Equation (A.1) becomes

$$\frac{1}{2\pi l_B^2}\iint \mathrm{d}\xi_x\,\mathrm{d}\xi_y\,\mathrm{e}^{-\frac{1}{2l_B^2}\bar{\xi}(\xi-z)} :V\left(\xi, 2l_B^2\frac{\partial}{\partial\xi}\right): F(\xi,z). \tag{A.7}$$

Since $F(\xi, z)$ was assumed analytic in ξ and $:V\left(\xi, 2l_B^2\frac{\partial}{\partial\xi}\right):$ is an operator on the Hilbert space of analytic functions (Bargmann space), the resulting function $:V\left(\xi, 2l_B^2\frac{\partial}{\partial\xi}\right): F(\xi, z)$ is an analytic function $f(\xi)$ in ξ and the integral (A.7) reduces to Bargmann's integral (A.2), which applied gives equation (A.4).

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